Fractons, strange correlators and dualities via measuring cluster states

Based on a work in preperation with
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Overview

Introduction
- Fractons
- Strange correlator
- Cluster states

Plaquette Ising model
- Non-invertible fusion rules
- Strange correlator

Generalization to CSS codes

Examples: Fractons
- X-cube model
- Checker board model
- Haah's code

Conclusion

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- Potential candidate for building self-correcting quantum memory.
- We are yet to find a broad framework which describes the fracton phases of matter.

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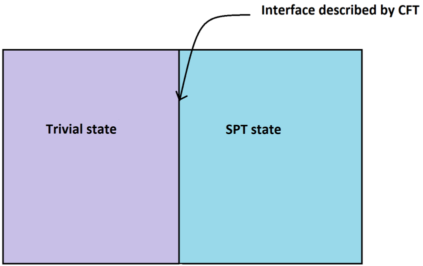
- An intuitive picture



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- In this talk, we will look at a similar concept for some examples of fracton orders.

Notation

Cell-complex:

Symbol	Meaning	Examples
σ_i	elementary cell	vertex, edge, plaquette.
σ_i^*	elementary dual-cell	vertex, edge, plaquette on the dual lattice.
Δ_i/Δ_i^*	set of cells/dual-cells	set of vertices on the lattice. set of vertices on the dual-lattice.
$\|\psi\rangle^{\Delta_i}$	$\bigotimes_{\sigma_i \in \Delta_i} \|\psi\rangle$	$\|+\rangle^{\Delta_v} = \bigotimes_{\sigma_v \in \Delta_v} \|+\rangle$.
C_i/C_i^*	Chain/dual chain which is formal linear combination of σ_i	C_v, C_e or C_v^*, C_e^*
c_i/c_i^*	A particular chain/ cochain	$c_v = \sigma_{v_1} + \sigma_{v_2}$ for v_1, v_2 two vertices. $c_v^* = \sigma_{v_1}^* + \sigma_{v_2}^*$ for v_1^*, v_2^* dual vertices.
$X(c)$	$X(c) = \prod_{\sigma_i \in c_i} X(\sigma_i)$	$X(c_v) = X(\sigma_{v_1})X(\sigma_{v_2})$.
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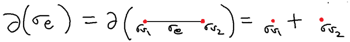
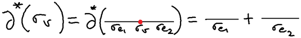
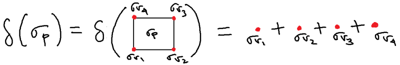
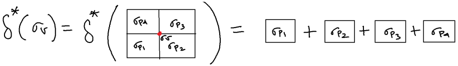
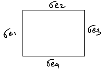
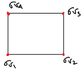
- Pauli-x, Pauli-y and Pauli-z operators denoted by X, Y and Z respectively.

Notation

Boundary map:

∂	Usual boundary operator	
∂^*	Usual coboundary operator	
δ	Unusual boundary operator	
δ^*	Unusual coboundary operator	
z	cycle, $\partial z = 0, \delta z = 0$	 $z = \sigma_{e_1} + \sigma_{e_2} + \sigma_{e_3} + \sigma_{e_4}, \ \partial z = 0.$
z^*	dual cycle, $\partial^* z^* = 0, \delta^* z^* = 0$	 $z^* = \sigma_{v_1} + \sigma_{v_2} + \sigma_{v_3} + \sigma_{v_4}, \ \partial^* z^* = 0.$

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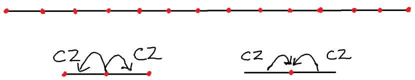
- Performing measurements on them can produce long range entangled states. [Rausendorff et al.[3]],[Tandivasadakrn et al[4]]
- As an example consider the 1d lattice consisting of vertices and edges. The cluster state is defined as

$$|\Psi_{\mathcal{C}}\rangle = \prod_{e\in\Delta_e}\prod_{v\in\partial e} CZ_{v,e}\,|+\rangle^{\Delta_v}\,|+\rangle^{\Delta_e}\,, \quad CZ_{v,e} = |0\rangle_v\langle 0| + |1\rangle_v\langle 1| \otimes Z_e \tag{3}$$



Stabilized by

$$X(\sigma_e)\prod_{v\in\partial e} Z(\sigma_v), \quad X(\sigma_v)\prod_{e\in\partial^* v} Z(\sigma_e) \tag{4}$$

Cont

- Performing measurements on the vertices in X basis with post-selection ${}^{\Delta_v}\langle+|\,|\Psi_{\mathcal{C}}\rangle$, gives the GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0...0\rangle + |1...1\rangle\right) \tag{5}$$

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- We will look at this idea with examples mainly focusing on fractons.

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- Hamiltonian

$$H_{qPIM} = -\sum_{\sigma_{i,j}} Z(\sigma_{i,j})Z(\sigma_{i,j+1})Z(\sigma_{i+1,j})Z(\sigma_{i+1,j+1}) \qquad (6)$$

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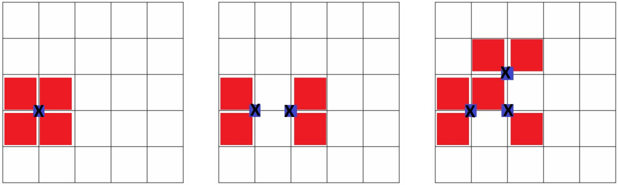
- The excitations can move only along lines and hence are fractons.



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With transverse field

- Now we add a transverse field to this Hamiltonian

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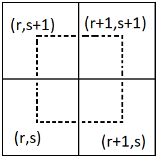

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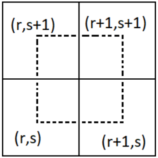

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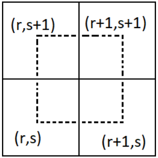

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- This is analogous to the story we are familiar with 1d transverse field Ising model.

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- This establishes the KW duality between H_{TFPI} and $\tilde{H}_{TFPI}$ using measurements.

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  give the partition function of classical plaquette Ising model.

- This is the overlap between fractonic ground state and a product state. Hence it can be interpreted as strange correlator.

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Examples: Fractons

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Haah's code

Conclusion

CSS codes

- A general CSS code can be represented by the following chain complex

$$0 \to C_Z \underset{\delta_Z^*}{\overset{\delta_Z}{\rightleftharpoons}} C_q \underset{\delta_X^*}{\overset{\delta_X}{\rightleftharpoons}} C_X \to 0 \tag{20}$$

  C_Z, C_q and C_X are free abelian groups generated by cells in Δ_Z, Δ_q and Δ_X [Kubica,Yoshida[5]]. They can also be thought of as vector space with $\mathbb{F}_2$ (characteristic 2 field) coefficients.

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- CSS code Hamiltonian

$$\mathbf{H}_{CSS} = -\sum_{\sigma_\beta \in \Delta_Z} Z(\delta_Z \sigma_\beta) - \sum_{\sigma_\alpha \in \Delta_X} X(\delta_X^* \sigma_\alpha) \tag{21}$$

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- Consider $H_q = \frac{\text{Ker } \delta_X}{\text{Im } \delta_Z}$ and $H^q = \frac{\text{Ker } \delta_Z^*}{\text{Im } \delta_X^*}$. Logical operators

$$Z(z_q) \text{ s.t } z_q \in H_q \qquad\qquad X(z_q^*) \text{ s.t } z_q \in H^q \tag{22}$$

With transverse field

- Hamiltonian with transverse field

$$H_{CSS} = -\sum_{\sigma_i \in \Delta_\mathrm{q}} X(\sigma_i) - \lambda \sum_{\sigma_\beta \in \Delta_Z} Z(\delta_Z \sigma_\beta) \quad \text{with symmetry } X(z_q^*)(\delta_Z^* z_q^* = 0) \tag{23}$$

$$H_{\mathrm{CSS,dual}} = -\sum_{\sigma_i \in \Delta_\mathrm{q}} Z(\delta_Z^* \sigma_i) - \lambda \sum_{\sigma_\beta \in \Delta_Z} X(\sigma_\beta) \quad \text{with symmetry } X(z_Z)(\delta_Z z_Z = 0) \tag{24}$$

With transverse field

- Hamiltonian with transverse field

$$H_{CSS} = -\sum_{\sigma_i \in \Delta_q} X(\sigma_i) - \lambda \sum_{\sigma_\beta \in \Delta_Z} Z(\delta_Z \sigma_\beta) \quad (23)$$

$$\text{with symmetry } X(z_q^*)(\delta_Z^* z_q^* = 0)$$

$$H_{\text{CSS,dual}} = -\sum_{\sigma_i \in \Delta_q} Z(\delta_Z^* \sigma_i) - \lambda \sum_{\sigma_\beta \in \Delta_Z} X(\sigma_\beta) \quad (24)$$

$$\text{with symmetry } X(z_Z)(\delta_Z z_Z = 0)$$

- As an example consider $0 \to C_2 \underset{\partial_2^*}{\overset{\partial_2}{\rightleftharpoons}} C_1 \underset{\partial_0^*}{\overset{\partial_0}{\rightleftharpoons}} C_0 \to 0$

$$H_{\text{gauge}} = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1) - \lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial_2 \sigma_2) \quad (25)$$

$$\text{with symmetry } X(z_1)(\partial_2^* z_1 = 0)\text{including the gauge symmetry } X(\partial_0^* \sigma_0)$$

$$H_{\text{Ising}} = -\sum_{\sigma_1 \in \Delta_1} Z(\partial_2^* \sigma_1) - \lambda \sum_{\sigma_2 \in \Delta_2} X(\sigma_2) \quad (26)$$

$$\text{with symmetry } \prod_{\sigma_2 \in \Delta_2} X(\sigma_2)$$

This is the Ising model-Ising gauge theory duality.

Kramers-Wannier transformation

- $\mathsf{KW} = \langle + |^{\otimes \Delta_{\mathrm{q}}} \mathcal{U}_{CZ} | + \rangle^{\otimes \Delta_Z}$ and $\mathsf{KW}' = \langle + |^{\otimes \Delta_Z} \mathcal{U}_{CZ} | + \rangle^{\otimes \Delta_{\mathrm{q}}}$.

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- Non-invertible fusion rules:

$$\mathsf{KW} \circ \mathsf{KW}' = \frac{1}{2^{|\Delta_Z|}} \sum_{\delta_Z z_Z = 0} X(z_Z) \tag{27}$$

$$\mathsf{KW}' \circ \mathsf{KW} = \frac{1}{2^{|\Delta_\mathrm{q}|}} \sum_{\delta_Z^* z_\mathrm{q}^* = 0} X(z_\mathrm{q}^*) \tag{28}$$

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- Strange correlator:

$$\mathsf{Z}_{\mathrm{sub}} = \mathcal{N} \times \langle \omega(K) | \mathsf{KW}' | + \rangle^{\Delta_Z} , \tag{29}$$

with $\langle \omega(K) | = \bigotimes_{\sigma_i \in \Delta_q} \langle 0 | e^{K X_{\sigma_i}}$.

$$\mathsf{Z}_{\mathrm{sub}} = \sum_{\{s(\sigma_\beta) = \pm 1\}_{\sigma_\beta \in \Delta_Z}} \exp\left[K \sum_{\sigma_i \in \Delta_{\mathrm{q}}} s(\delta_Z^* \sigma_i) \right] \tag{30}$$

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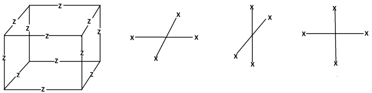
X-cube model

- Hamiltonian

$$\mathbf{H}_{X-cube} = -\sum_{c} \prod_{e \in \partial c} Z_e - \sum_{v} \prod_{e \in \delta_1^* v} X_e - \sum_{v} \prod_{e \in \delta_2^* v} X_e - \sum_{v} \prod_{e \in \delta_3^* v} X_e \qquad (31)$$



X-cube model

- Hamiltonian

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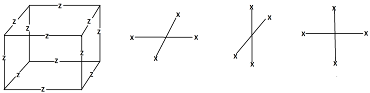

- Hamiltonian with transverse field

$$H_{X-cube} = -\sum_{c}\prod_{e\in\partial c} Z_e - \sum_{e} X_e \text{ with subsytem one form symmetry} \quad (32)$$

$$H_{3d-qPIM} = -\sum_{p}\prod_{v\in\partial p} Z_v - \sum_{v} X_v \text{ with subsystem plane symmetry} \quad (33)$$

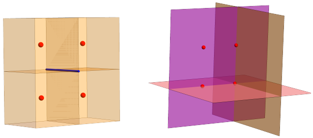

Figure 2: The 3d classical plaquette Ising model and its global symmetry.

- $KW = \langle + |^{\Delta_e} \mathcal{U}_{CZ}^{X-cube} | + \rangle^{\Delta_c}$ and $KW' = \langle + |^{\Delta_c} \mathcal{U}_{CZ}^{3d-qPIM} | + \rangle^{\Delta_e}$

- $KW = \langle + |^{\Delta_e} \mathcal{U}_{CZ}^{X-cube} | + \rangle^{\Delta_c}$ and $KW' = \langle + |^{\Delta_c} \mathcal{U}_{CZ}^{3d-qPIM} | + \rangle^{\Delta_e}$
- Fusion rules:

$$\begin{aligned} KW' \circ KW &= \sum \text{subsystem one form symmetries} \\ KW \circ KW' &= \sum \text{subsystem plane like symmetries} \end{aligned} \tag{34}$$

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- Strange correlator:

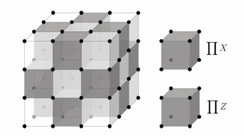
$$\begin{aligned} Z_{3d-cPIM} &= \mathcal{N} \langle \omega(K) | \, KW' \, | + \rangle^{\Delta_c} \\ &= \sum_{s_{\sigma_c} = \pm 1} e^{K \sum_{\sigma_e \in \Delta_e} s(\delta^* \sigma_e)} \end{aligned} \tag{35}$$

Partition function of 3d classical plaquette Ising model.

Checkerboard model

- Hamiltonian

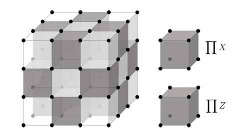
$$\mathbf{H}_{check} = -\sum_{c^{(s)}} \prod_{v \in \partial c^{(s)}} Z_v - \sum_{c^{(s)}} \prod_{v \in \partial c^{(s)}} X_v \quad (36)$$



Checkerboard model

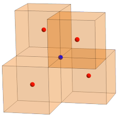
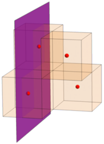
- Hamiltonian

$$\mathbf{H}_{check} = -\sum_{c^{(s)}} \prod_{v \in \partial c^{(s)}} Z_v - \sum_{c^{(s)}} \prod_{v \in \partial c^{(s)}} X_v \qquad (36)$$



- Hamiltonian with transverse field

$$\begin{aligned} H_{check} &= -\sum_{c^{(s)}} \prod_{v \in \partial c^{(s)}} Z_v - \sum_{v} X_v \text{ with subsystem line symmetry} \\ H_{tIM} &= -\sum_{v} \prod_{c^{(s)} \in \delta^*_Z v} Z_{c^{(s)}} - \sum_{c^{(s)}} X_{c^{(s)}} \text{ with subsystem plane symmetry} \end{aligned} \qquad (37)$$

- $KW = \langle + |^{\Delta_{c(s)}} \mathcal{U}_{CZ}^{check} | + \rangle^{\Delta_v}$ and $KW' = \langle + |^{\Delta_v} \mathcal{U}_{CZ}^{tIM} | + \rangle^{\Delta_{c(s)}}$

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- Fusion rules:

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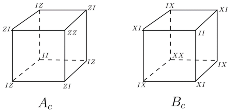
$$\begin{aligned} Z_{tIM} &= \mathcal{N} \langle \omega(K)| KW' |+\rangle^{\Delta_{c(s)}} \\ &= \sum_{s_{\sigma_{c(s)}}=\pm 1} e^{K \sum_{\sigma_v \in \Delta_v} s(\delta^* \sigma_v)} \end{aligned} \tag{39}$$

Partition function of tetrahedral Ising model.

Haah's code

- Hamiltonian

$$\mathbf{H}_{Haah} = \sum_c A_c - \sum_c B_c \tag{40}$$



Haah's code

- Hamiltonian

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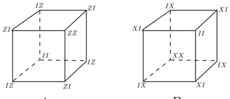
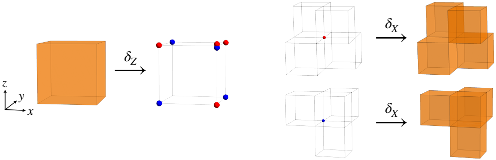


A_c B_c

- Hamiltonian with transverse field

$$H_{Haah} = -\sum_c A_c - \sum_{v_1} X_{v_1} - \sum_{v_2} X_{v_2} \tag{41}$$

$$H_{fIM} = -\sum_{v_1} \prod_{c\in\partial^* v_1} Z_c - \sum_{v_2} \prod_{c\in\partial^* v_2} Z_c - \sum_c X_c \tag{42}$$



- $KW = \langle + |^{\Delta_{v_1 \cup v_2}} \mathcal{U}_{CZ}^{Haah} | + \rangle^{\Delta_c}$ and $KW' = \langle + |^{\Delta_c} \mathcal{U}_{CZ}^{fIM} | + \rangle^{\Delta_{v_1 \cup v_2}}$

- $KW = \langle + |^{\Delta_{v_1 \cup v_2}} \mathcal{U}_{CZ}^{Haah} | + \rangle^{\Delta_c}$ and $KW' = \langle + |^{\Delta_c} \mathcal{U}_{CZ}^{fIM} | + \rangle^{\Delta_{v_1 \cup v_2}}$
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$$\begin{aligned} KW' \circ KW &= \sum \text{subsystem fractal symmetries} \\ KW \circ KW' &= \sum \text{subsystem fractal symmetries} \end{aligned} \tag{43}$$

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Partition function of fractal Ising model.

Generalized Ising-gauge theory duality

- So we found classical spin models as Strange correlator of fracton ground state with a product state.

Generalized Ising-gauge theory duality

- So we found classical spin models as Strange correlator of fracton ground state with a product state.
- This agrees with the duality of fracton orders with spin models under generalized lattice gauge theory duality [Vijay,Haah,Fu[6]].

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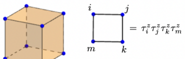
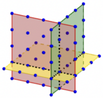
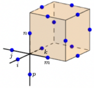
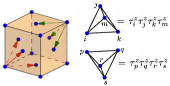
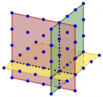
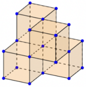
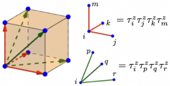
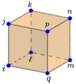
-

Spin system	Subsystem symmetry gauged	Fracton model
3d Plaquette Ising model (3d-qPIM)	Planar symmtery	X-cube model
Tetrahedral Ising model (tIM)	Planar symmtery	Checker board model
Fractal Ising model (fIM)	Fractal symmetry	Haah's code

Chart

[Vijay,Haah,Fu[6]]

Classical Spin System	Subsystem Symmetry	Fracton Topological Phase
 $= \tau_i^z\tau_j^z\tau_k^z\tau_m^z$ **Plaquette Ising Model**	 Planar	 $A_c = \prod_{n\in\partial c}\sigma_n^x$ $B_s^{(xz)} = \sigma_i^z\sigma_p^z\sigma_n^z\sigma_k^z$ $B_s^{(xy)} = \sigma_i^z\sigma_j^z\sigma_k^z\sigma_m^z$ $B_s^{(yz)} = \sigma_j^z\sigma_n^z\sigma_m^z\sigma_p^z$ **X-Cube Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(1)},\ m_b^{(1)}\right]$
 $= \tau_i^z\tau_j^z\tau_k^z\tau_m^z$ $= \tau_p^z\tau_q^z\tau_r^z\tau_s^z$ **Tetrahedral Ising Model**	 Planar	 $A_c = \prod_{n\in\partial c}\sigma_n^x$ $B_c = \prod_{n\in\partial c}\sigma_n^z$ **Checkerboard Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(0)}\right]$
 $= \tau_i^z\tau_j^z\tau_k^z\tau_m^z$ $= \tau_i^z\tau_p^z\tau_q^z\tau_r^z$ **Fractal Ising Model**	 Fractal	 $A_c = \mu_i^z\sigma_j^z\mu_k^z\mu_m^z\sigma_n^z\mu_p^z\sigma_p^z\sigma_q^z$ $B_c = \mu_i^x\sigma_j^x\mu_k^x\sigma_\ell^x\mu_\ell^x\mu_m^x\sigma_n^x\mu_p^x\sigma_q^x$ **Haah's Code** $\left[\text{Type II}:\ e_a^{(0)},\ m_a^{(0)}\right]$

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Conclusion and future directions

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Future directions:

- Strange correlator interpretation for fermion.
- Is there a deep understanding for strange correlator in the case of fractonic phases of matter.
- Potential application of strange correlator to fractonic systems.

List of References I

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Thank You!